Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam
August 2015: Problem 4 Solution

Exercise.

(a) Let $[a, b]$ be a closed, bounded interval and $f : [a, b] \to \mathbb{R}$. Give an "epsilon-delta definition" of what it means for f to be "absolutely continuous on $[a, b]$".

Solution.

f is **absolutely continuous on** $[a, b]$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ s.t. $(a_j, b_j) \subseteq [a, b]$ for all j,

$$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^n |f(b_j) - f(a_j)| < \epsilon$$

(b) Assume now that $f[0, 1] \to \mathbb{R}$ has the property that for every ϵ such that $0 < \epsilon < 1$, the restriction of f to the closed interval $[\epsilon, 1]$ is absolutely continuous. Assume also that there exists some $p > 2$ such that and that

$$\int_0^1 x|f'(x)|^p dm < \infty,$$

where m denotes Lebesgue measure. Prove that $\lim_{x \to 0} f(x)$ exists and is finite.

Solution.

We will use **The Fundamental Theorem of Calculus for Lebesgue Integrals:**
If $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{C}$, the following are equivalent

(a) f is absolutely continuous on $[a, b]$

(b) $f(x) - f(a) = \int_a^x f^*(t)dt$ for some $f^* \in L^1([a, b], m)$

(c) f is differentiable a.e. on $[a, b]$, $f' \in L^1([a, b], m)$ and $f(x) - f(a) = \int_a^x f'(t)dt$

If $f\Big|_{[\epsilon,1]}$ is absolutely continuous for every ϵ such that $0 < \epsilon < 1$, then f is differentiable a.e. on $[\epsilon, 1]$ for $0 < \epsilon < 1$

$\implies$ f is differentiable a.e. on $[0, 1]$

Also, for $0 < \epsilon < 1$,

$$f(1) - f(\epsilon) = \int_\epsilon^1 f'(t)dt \qquad \implies \qquad f(\epsilon) = f(1) - \int_\epsilon^1 f'(t)dt$$

we want to show $f' \in L^1([0, 1], m)$.

Solution.

$$
\begin{aligned}
\int_0^1 |f'(x)|dm &= \int_0^1 |x(f'(x))^p|^{\frac{1}{p}}|x^{-\frac{1}{p}}|dm \\
&= \int_0^1 x^{\frac{1}{p}}|f'(x)| \cdot x^{-\frac{1}{p}}dm \\
&\leq \left[\int_0^1 \left|x^{\frac{1}{p}}f'(x)\right|^p dm\right]^{\frac{1}{p}} \cdot \left[\int_0^1 x^{-\frac{1}{p}\cdot\frac{p}{p-1}}dm\right]^{\frac{p-1}{p}} \qquad \text{by Holders inequality} \\
&= \left[\int_0^1 x\,|f'(x)|^p\,dm\right]^{\frac{1}{p}} \cdot \left[\int_0^1 x^{-\frac{1}{p-1}}dm\right]^{\frac{p-1}{p}}
\end{aligned}
$$

$$
\left[\int_0^1 x\,|f'(x)|^p\,dm\right]^{\frac{1}{p}} < \infty \qquad \text{and} \qquad p > 2 \implies p-1 > 1 \implies 1 > \frac{1}{p-1}
$$

$$
\implies \qquad \int_0^1 x^{-\frac{1}{p-1}}dx < \infty \qquad \text{since} \qquad x^{-\frac{2}{p-1}} < \infty \text{ is integrable for } x \in [0,1]
$$

$$
\implies \qquad \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(f(1) - \int_x^1 f'(u)du\right) = f(1) - \int_0^1 f'(u)du < \infty
$$